A COMPLETELY INTEGRABLE SYSTEM ON G_2 COADJOINT ORBITS

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Abstract. We construct a Gelfand-Zeitlin system on a one-parameter family of G_2 coadjoint orbits that are multiplicity-free Hamiltonian $SU(3)$-spaces. Using this system we prove a lower bound for the Gromov width of these orbits. This lower bound agrees with the known upper bound.

1. Introduction

Gelfand-Zeitlin systems are completely integrable systems constructed from maximal non-Abelian Hamiltonian symmetries of a symplectic manifold. The classic examples of Gelfand-Zeitlin systems were constructed on $SU(n)$ and $SO(n)$ coadjoint orbits [6]. These systems were employed by [16] to prove tight lower bounds for the Gromov width of $SU(n)$ and $SO(n)$ coadjoint orbits that, together with the upper bounds obtained in [4], compute the Gromov width. Tight lower bounds were also recently proven for $Sp(n)$ coadjoint orbits using integrable systems constructed from toric degenerations [8].

Motivated by the work of [16], and a desire for more examples of Gelfand-Zeitlin systems, this paper describes a Gelfand-Zeitlin system on a one-parameter family of G_2 coadjoint orbits that are multiplicity-free $SU(3)$-spaces. As complex manifolds, these coadjoint orbits are degree 18, 5 dimensional projective subvarieties of $\mathbb{CP}^{13}$ [1]. Using the convexity theorem of [13] we are able to compute the image of the Gelfand-Zeitlin system – a 5-dimensional convex polytope – without undue difficulty and this image classifies an open dense submanifold of the coadjoint orbit up to symplectomorphism. Combining the lower bound methods employed recently in [16], and the upper bounds due to [4], we can then prove

Theorem 1. *If $\mathcal{O}_\lambda$ is the G_2 coadjoint orbit through $(0,\lambda)$ as in Figure 1 equipped with the Kostant-Kirillov-Souriau symplectic form ω_λ, then*

$$GWidth(\mathcal{O}_\lambda,\omega_\lambda)=\frac{\lambda}{\sqrt{3}}.$$

The other one-parameter family of 5 (complex) dimensional G_2 coadjoint orbits, corresponding to the short simple root, are quadrics in $\mathbb{CP}^6$ that are homogeneous spaces for $SO(7,\mathbb{C})$. As such, they are $SO(8)$ coadjoint orbits and carry Gelfand-Zeitlin systems already described in [16] that, combined with the upper bound of [4], give the precise Gromov width of these orbits.

One may also deduce the Gromov width of both these families of 5 dimensional G_2 coadjoint orbits from the recent work of [14], since they have second Betti number $b_2 = 1$. The lower bound in the main theorem of *op. cit.* follows from work of [9, 12] that constructs completely integrable torus actions on projective varieties from toric degenerations. In comparison, the virtue of this work is that the momentum map and polytope have a very straightforward descriptions.

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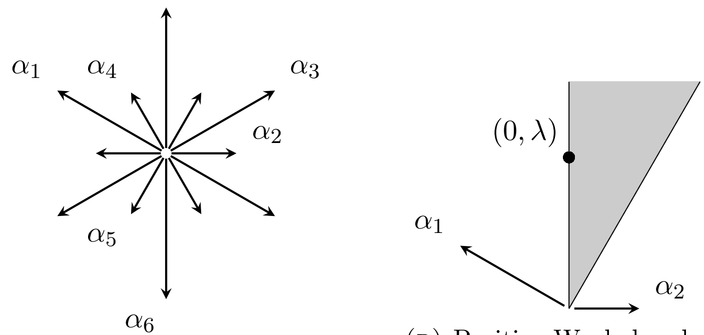


(A) Root system of G_2

(B) Positive Weyl chamber of G_2

FIGURE 1

2. Gelfand-Zeitlin systems and multiplicity-free spaces

Introduced by [6], Gelfand-Zeitlin systems can be viewed as examples of the following general construction, a detailed exposition of which can be found in [13].

Let G be a compact, connected Lie group and let (M, ω, Φ, G) be a connected Hamiltonian G-space. Given a chain of subgroups $H_1 \leq \cdots \leq H_n = G$ with corresponding subalgebras $\mathfrak{h}_k = \mathrm{Lie}(H_k)$, there are dual projection maps $p_k : \mathfrak{h}_{k+1}^* \to \mathfrak{h}_k^*$. Fixing closed positive Weyl chambers, $\mathfrak{t}_{k,+}^* \subset \mathfrak{h}_k^*$, we can construct the map

$$\begin{aligned} &\Lambda : \mathfrak{g}^* \to \mathcal{T} = \mathfrak{t}_{n,+}^* \times \cdots \times \mathfrak{t}_{1,+}^* \subset \mathfrak{t}_n^* \times \cdots \times \mathfrak{t}_1^* \\ &\Lambda = (s, s \circ p_{n-1}, \ldots, s \circ p_1 \circ \cdots \circ p_{n-1}) \,. \end{aligned} \tag{2}$$

where $s(\xi)$ is defined as the unique element of $(H_k \cdot \xi) \cap \mathfrak{t}_{k,+}^*$. The restriction of the map

$$\Lambda \circ \Phi : M \to \mathcal{T} \tag{3}$$

to the open, dense submanifold $U = (\Lambda \circ \Phi)^{-1}(\mathcal{T}^{\mathrm{int}})$ is a smooth momentum map for a Hamiltonian torus action[1]. This construction is referred to as *Thimm's trick* or *the method of Thimm*, and one may say that the map $\Lambda \circ \Phi$ is *constructed by Thimm's trick,* in reference to the paper [18].

Recall that a Hamiltonian G-space is *multiplicity-free* if for all $\xi \in \mathfrak{g}^*$ the reduced space $\Phi^{-1}(\xi)/G_\xi$ is a point. An effective Hamiltonian torus action is completely integrable if the dimension of the torus is half the real dimension of the manifold. If M is a multiplicity-free G-space, and for each $1 \leq k < n$, every H_{k+1} coadjoint orbit is a multiplicity-free H_k-space, the Hamiltonian torus action induced on U is completely integrable [7]. Following Guillemin and Sternberg's terminology in [6], we refer to such a map (and the associated densely defined torus action) as a *Gelfand-Zeitlin*[2] *system.*

[1] To be more precise, one should replace each of the closed chambers $\mathfrak{t}_{k,+}^*$ in (2) with the closure of the principal face of $\mathfrak{t}_{k,+}^*$ corresponding to the induced Hamiltonian H_k-actions on M.

[2] One should note that there are multiple spellings of Zeitlin in the literature, including Tsetlin and Cetlin. Works by Kostant-Wallach, Kogan-Miller, and Guillemin-Sternberg respectively each choose a different spelling.

Example 4. Let $\mathcal{O}$ be a $U(k)$ coadjoint orbit, and define the subgroup

$$H_{k-1} = \{\mathrm{diag}\,(A, 1) : A \in U(k-1)\} \cong U(k-1).$$

Guillemin and Sternberg showed that $\mathcal{O}$ is a multiplicity-free $U(k-1)$-space. Thus if M is a multiplicity-free $SU(n)$-space, then the map $\Lambda \circ \Phi$ constructed on M by Thimm's trick for the chain of subgroups

$$U(1) \leq U(2) \leq \cdots \leq U(n-1) \leq SU(n)$$

is a Gelfand-Zeitlin system.

If $\Lambda \circ \Phi : M \to \mathcal{T}$ is a map constructed by Thimm's trick, then the submanifold U is connected. Using this fact and the convexity theorem for proper torus momentum maps [2, 3] one can prove

Theorem 5. [13] *Let M be a connected symplectic manifold and suppose $\Lambda \circ \Phi : M \to \mathcal{T}$ is a map constructed by Thimm's trick. If $\Lambda \circ \Phi$ is a proper map, then $\Lambda \circ \Phi(M)$ is convex and the fibres of $\Lambda \circ \Phi$ are connected.*

3. A Gelfand-Zeitlin system on $\mathcal{O}_\lambda$

In [20], Chris Woodward gives the following example.

Example 6. Consider the exceptional Lie group G_2 with maximal torus T and let $\mathfrak{t}$ be the corresponding subalgebra. Fix the standard identifications $\mathfrak{t} \cong \mathfrak{t}^* \cong \mathbb{R}^2$ and let

$$S = \left\{ \alpha_1 = \left(\frac{-3}{2}, \frac{\sqrt{3}}{2} \right), \alpha_2 = (1, 0) \right\} \subset \mathbb{R}^2$$

be the standard choice of simple roots (see Figure 1A). The positive Weyl chamber $\mathfrak{t}^*_{G_2,+}$ given by this choice of simple roots is shaded in Figure 1B. There is an embedding of $SU(3)$ in G_2 with maximal torus T corresponding to the long roots of G_2, as shown in Figure 2A.

Coadjoint G_2-orbits are parameterized by points in $\mathfrak{t}^*_{G_2,+}$. In particular, we write $\mathcal{O}_\lambda$ for the G_2 coadjoint orbit through the point $(0, \lambda)$ on the vertical edge of $\mathfrak{t}^*_{G_2,+}$, equipped with its Kostant-Kirillov-Souriau symplectic form ω_λ. The following facts are explained in [20].

(1) The coadjoint orbit $\mathcal{O}_\lambda \cong G_2/(SU(2) \times U(1))$ has real dimension 10.
(2) The $SU(3)$ subgroup acts locally freely on $\mathcal{O}_\lambda$. Since $\dim SU(3) + \mathrm{rank} SU(3) = 10$, $\mathcal{O}_\lambda$ is a multiplicity free $SU(3)$-space.
(3) The Kirwan polytope for the $SU(3)$-action can be understood using the Heckman formula of [5] and is the triangle illustrated in Figure 2B. In coordinates, the three vertices of this triangle are

$$(0, \lambda), \left(\frac{\lambda}{2\sqrt{3}}, \frac{\lambda}{2} \right), \left(\frac{-\lambda}{2\sqrt{3}}, \frac{\lambda}{2} \right).$$

In terms of the roots of G_2 the vertices are

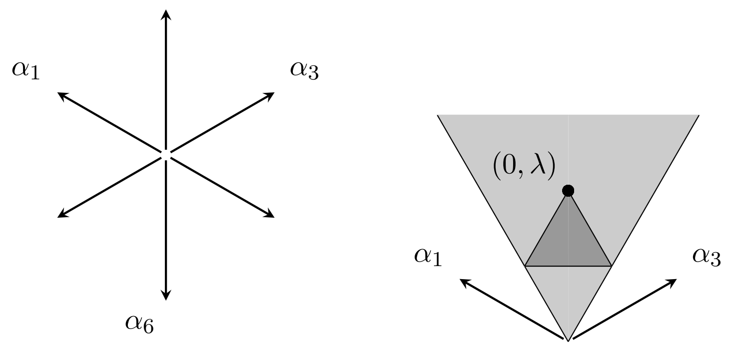
$$\frac{\lambda(\alpha_4 - \alpha_5)}{\sqrt{3}}, \frac{-\lambda\alpha_5}{\sqrt{3}}, \frac{\lambda\alpha_4}{\sqrt{3}}.$$

Following the previous section, consider the chain of groups

$$U(1) \leq U(2) \leq SU(3)$$

where we choose the embeddings

$$U(1) \cong \{\mathrm{diag}\,(a, 1) : a \in U(1)\} \leq U(2),\; U(2) \cong \left\{ \frac{1}{\det(A)} \mathrm{diag}\,(A, 1) : A \in U(2) \right\} \leq SU(3).$$



(A) Root system of $SU(3)$ (B) Kirwan polytope of $\mathcal{O}_\lambda$

FIGURE 2

Since $\mathcal{O}_\lambda$ is a multiplicity-free $SU(3)$-space, the map

$$\Lambda \circ \Phi : \mathcal{O}_\lambda \to \mathfrak{t}^*_{SU(3),+} \times \mathfrak{t}^*_{U(2),+} \times \mathfrak{t}^*_{U(1),+} \tag{7}$$

is a Gelfand-Zeitlin system. For the remainder of this paper $\Lambda \circ \Phi$ will denote this specific Gelfand-Zeitlin system.

4. THE GELFAND-ZEITLIN POLYTOPE OF $\mathcal{O}_\lambda$

In this section we compute the image of the Gelfand-Zeitlin system on $\mathcal{O}_\lambda$. In order to do this we compute the vertices of the image, a list of bounding inequalities, and apply Theorem 5.

Fix the standard identifications

$$\begin{aligned}
&\mathfrak{u}(n) \cong \left\{ X \in M_n\mathbb{C} : X = \overline{X}^t \right\} \\
&\mathfrak{t}^*_{U(n)} \cong \mathfrak{t}_{U(n)} \cong \{\mathrm{diag}(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\} \cong \mathbb{R}^n \\
&L^*_{U(n)} = \mathrm{Hom}(L_{U(n)}, \mathbb{Z}) \cong L_{U(n)} \cong \{\mathrm{diag}(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{Z}\} \cong \mathbb{Z}^n \\
&\mathfrak{t}^*_{U(n),+} \cong \{\mathrm{diag}(x_1, \dots, x_n) \in \mathfrak{t}_u(n) : x_1 \leq \dots \leq x_n\}
\end{aligned}$$

From the previous section we also have the identifications

$$\begin{aligned}
\mathfrak{t}^*_{SU(3)} &= \mathbb{R}^2 \\
L^*_{SU(3)} &= \mathbb{Z}\left\langle \alpha_2 = (1,0)\,,\ \alpha_4 = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), \right\rangle.
\end{aligned}$$

With these identifications the Gelfand-Zeitlin system (7) is a map to $\mathcal{T} \subset \mathbb{R}^5 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^1 \cong \mathfrak{t}^*_{SU(3)} \times \mathfrak{t}^*_{U(2)} \times \mathfrak{t}^*_{U(1)}$ with coordinates $(x_1, x_2, x_3, x_4, x_5)$, where

$$\begin{aligned}
\mathcal{T} &= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : \frac{\pm 3x_1}{2} + \frac{\sqrt{3}x_2}{2} \geq 0 \text{ and } x_3 \leq x_4 \right\} \\
&\cong \mathfrak{t}^*_{SU(3),+} \times \mathfrak{t}^*_{U(2),+} \times \mathfrak{t}^*_{U(1),+}
\end{aligned}$$

Further, in these coordinates the weight lattice $L^* = \mathrm{Hom}(L, \mathbb{Z})$ is

$$\begin{aligned}
L^* &= L^*_{SU(3)} \times L^*_{U(2)} \times L^*_{U(1)} \\
&= \mathbb{Z}\left\langle (1,0,0,0,0)\,, \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0\right), (0,0,1,0,0)\,, (0,0,0,1,0)\,, (0,0,0,0,1) \right\rangle.
\end{aligned} \tag{8}$$

TABLE 1. Vertices of the Gelfand-Zeitlin polytope Δ_λ

$(x_1, x_2, x_3, x_4, x_5)$
$(0, \lambda, 0, 0, 0)$
$\left(0, \lambda, \frac{-\lambda}{\sqrt{3}}, 0, \frac{-\lambda}{\sqrt{3}}\right)$
$\left(0, \lambda, \frac{-\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}}\right)$
$\left(0, \lambda, 0, \frac{\lambda}{\sqrt{3}}, 0\right)$
$\left(0, \lambda, 0, \frac{\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}}\right)$
$\left(0, \lambda, \frac{-\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}}\right)$
$\left(0, \lambda, \frac{-\lambda}{\sqrt{3}}, 0, 0\right)$
$\left(\frac{-2\lambda}{3\sqrt{3}}, \frac{\lambda}{3\sqrt{3}}, \frac{-2\lambda}{3\sqrt{3}}, \frac{-2\lambda}{3\sqrt{3}}, \frac{-2\lambda}{3\sqrt{3}}\right)$
$\left(\frac{-2\lambda}{3\sqrt{3}}, \frac{\lambda}{3\sqrt{3}}, \frac{-2\lambda}{3\sqrt{3}}, \frac{\lambda}{3\sqrt{3}}, \frac{\lambda}{3\sqrt{3}}\right)$
$\left(\frac{-2\lambda}{3\sqrt{3}}, \frac{\lambda}{3\sqrt{3}}, \frac{-2\lambda}{3\sqrt{3}}, \frac{\lambda}{3\sqrt{3}}, \frac{-2\lambda}{3\sqrt{3}}\right)$
$\left(\frac{-\lambda}{3\sqrt{3}}, \frac{2\lambda}{3\sqrt{3}}, \frac{-\lambda}{3\sqrt{3}}, \frac{-\lambda}{3\sqrt{3}}, \frac{-\lambda}{3\sqrt{3}}\right)$
$\left(\frac{-\lambda}{3\sqrt{3}}, \frac{2\lambda}{3\sqrt{3}}, \frac{-\lambda}{3\sqrt{3}}, \frac{2\lambda}{3\sqrt{3}}, \frac{-\lambda}{3\sqrt{3}}\right)$
$\left(\frac{-\lambda}{3\sqrt{3}}, \frac{2\lambda}{3\sqrt{3}}, \frac{-\lambda}{3\sqrt{3}}, \frac{2\lambda}{3\sqrt{3}}, \frac{2\lambda}{3\sqrt{3}}\right)$

The Kirwan polytope of the previous section gives us inequalities

$$\begin{aligned} &\frac{-\sqrt{3}x_1}{2} + \frac{x_2}{2} \leq \frac{\lambda}{2}, \\ &\frac{\sqrt{3}x_1}{2} + \frac{x_2}{2} \leq \frac{\lambda}{2}, \\ &\frac{\lambda}{2} \leq x_2. \end{aligned} \tag{9}$$

From the interlacing inequalities for eigenvalues of Hermitian matrices, one derives

$$\begin{aligned} &\frac{-x_1}{3} - \frac{x_2}{\sqrt{3}} \leq x_3 \leq \frac{2x_1}{3} \leq x_4 \leq \frac{-x_1}{3} + \frac{x_2}{\sqrt{3}} \\ &x_3 \leq x_5 \leq x_4. \end{aligned} \tag{10}$$

One can find elements of $\Phi(\mathcal{O}_\lambda)$ that make each of the inequalities strict, so the image of $\Lambda \circ \Phi$ contains the points listed in Table 1. By Theorem 5 and the inequalities (9) and (10) we have

Proposition 11. *The image of the Gelfand-Zeitlin system $\Lambda \circ \Phi : \mathcal{O}_\lambda \to \mathbb{R}^5$ is the convex hull of the 13 points listed in Table 1. This convex hull, which we will refer to as Δ_λ, is a 5-dimensional convex polytope.*

5. THE GROMOV WIDTH OF $\mathcal{O}_\lambda$

The *Gromov width* of a symplectic manifold is defined as

$$\mathrm{GWidth}(M, \omega) = \sup\left\{\pi r^2 > 0 : \exists \text{ a symplectic embeddding } B^{2n}(r) \to M\right\}$$

where $B^{2n}(r)$ is the open ball of radius r in $\mathbb{R}^{2n}$ equipped with the standard symplectic structure. Coadjoint orbits have the Kostant-Kirillov-Souriau symplectic structure and

their Gromov width has been a subject of much study, including work on complex Grassmannians by [11, 15] and later other families of coadjoint orbits [21, 16, 4, 8, 14]. In this section we follow the same approach as [16] to compute a lower bound for the Gromov width of $\mathcal{O}_\lambda$ from the polytope Δ_λ.

Suppose a connected symplectic manifold (M,ω) is equipped with a completely integrable action of a torus T generated by a momentum map

$$\mu : M \to \mathfrak{t}^*.$$

If there is an open, convex set $\mathcal{C} \subset \mathfrak{t}^*$ such that $\mu(M) \subset \mathcal{C}$ and the map $\mu : M \to \mathcal{C}$ is proper, then M is classified up to T-equivariant symplectomorphism by its image, $\mu(M)$ ([10], Proposition 6.5). In particular, the open submanifold $\mu^{-1}(\mu(M)^{\mathrm{int}})$, with the restricted symplectic structure, is symplectomorphic to $\mu(M)^{\mathrm{int}} \times T$ with the symplectic structure ω such that $p_1 : \mu(M)^{\mathrm{int}} \times T \to \mathfrak{t}^*$ is a momentum map for the torus action $t \cdot (\xi, s) = (\xi, ts)$. Thus, to obtain a lower bound on the Gromov width of M, it is sufficient to obtain a lower bound on the Gromov width of $\mu(M)^{\mathrm{int}} \times T$. A lower bound on the Gromov width of $\mu(M)^{\mathrm{int}} \times T$ can be extracted from the the combinatorics of $\mu(M)$, as we will now explain. Define

$$\Delta^n(l) = \{(y_1, \dots, y_n) \in \mathbb{R}^n : 0 < y_1, \dots, y_n, \text{ and } y_1 + \cdots + y_n < l\},$$

the standard open simplex of size l. A version of the following Lemma was proven in [19] for $n = 2$. Proofs for arbitrary n can be found in [16, 17].

Recall that if (V_i, L_i), $i = 1, 2$, are vector spaces V_i together with full rank lattices L_i, then an *integral affine map* $B : V_1 \to V_2$ is a map of the form $B = A + v$ where $A : V_1 \to V_2$ is a linear map that sends a $\mathbb{Z}$-basis for L_1 to a $\mathbb{Z}$-basis for L_2 and $v \in V_2$.

Lemma 12. *Let T be a compact torus of dimension n and let S be an open subset of $\mathfrak{t}^*$. If there is an integral affine map $B : (\mathbb{R}^n, \mathbb{Z}^n) \to (\mathfrak{t}^*, L^*)$ such that $B(\Delta^n(l)) \subset S$, then*

$$l \leq GWidth(S \times T, \omega).$$

If a line segment in $(\mathbb{R}^n, L)$ can be translated to a scalar multiple of a primitive vector in the lattice L, then that scalar is the *integral affine length* of the line segment. A vertex v of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ is *smooth* if the edges of $\mathcal{P}$ incident to v are spanned by a $\mathbb{Z}$-basis for L (after translation by v). If v is a smooth vertex and the minimum integral affine length of the edges incident to v is l, then there exists an integral affine map B such that $B(\Delta^n(l)) \subset \mathcal{P}^{\mathrm{int}}$.

The other half of Theorem 1 follows from

Theorem 13. [4] *If $\mathcal{O}_\lambda$ is the G_2 coadjoint orbit through $(0, \lambda)$ as in Figure 1, then*

$$GWidth(\mathcal{O}_\lambda, \omega_\lambda) \leq \frac{\lambda}{\sqrt{3}}.$$

We are now equipped to prove the main theorem.

of Theorem 1.1. Recall that the pre-image of $\mathcal{T}^{\mathrm{int}}$ under $\Lambda \circ \Phi$ is a connected open dense submanifold $U \subset \mathcal{O}_\lambda$. The restriction

$$\Lambda \circ \Phi : U \to \mathcal{C} = \mathcal{T}^{int}$$

is a proper map to an open convex set that generates a completely integrable Hamiltonian torus action. Thus by [10], (U, ω_λ) is the unique non-compact toric manifold with moment polytope $\Lambda \circ \Phi(U) = \Delta_\lambda \cap \mathcal{C} \subset (\mathbb{R}^5, L^*)$.

It is straightforward to check from Table 1 that the minimum integral affine length of an edge of Δ_λ (with respect to the weight lattice L^* in Equation 8) is $\frac{\lambda}{\sqrt{3}}$. Any vertex of Δ_λ contained in $\mathcal{C}$ (of which there are several) is smooth, so by the discussion following Lemma

12 there exists an integral affine map $B : (\mathbb{R}^5, \mathbb{Z}^5) \to (\mathbb{R}^5, L^*)$ that embeds $\Delta^5\left(\frac{\lambda}{\sqrt{3}}\right)$ into $\Delta_\lambda^{\text{int}}$. Thus by Lemma 12 and the discussion preceding it we have shown that

$$\frac{\lambda}{\sqrt{3}} \leq \mathrm{GWidth}(\Delta_\lambda^{\text{int}} \times T, \omega) \leq \mathrm{GWidth}(\mathcal{O}_\lambda, \omega_\lambda).$$

Combining this with the upper bound of [4] completes the proof. □

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